Math 279 Lecture 12 Notes

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1 Exponential Martingale Bounds and Geometricity of the Stratonovich Integral

1.1 Exponential martingale methods for bounding Brownian motion increments

Our purpose is showing that our candidates

$$\mathbb{B}(s,t) = \lim_{n\to\infty} \underbrace{\sum_{t_i^n \in [s,t]} B(t_i^n) \otimes B(t_i^n, t_{i+1}^n)}_{\mathbb{B}_n(s,t)} - B(s) \otimes B(s,t),$$

$$\widehat{\mathbb{B}}(s,t) = \lim_{n\to\infty} \underbrace{\sum_{t_i^n \in [s,t]} \frac{B(t_i^n) + B(t_{i+1}^n)}{2} \otimes B(t_i^n, t_{i+1}^n)}_{\widehat{\mathbb{B}}_n(s,t)} - B(s) \otimes B(s,t)$$

Last time, we worked out the "quadratic variation" of $\mathbb{B}_n$ and applied the Burkholder-Davis-Gundy inequality to get the desired bound. Alternatively, we can use the so-called exponential martingale to get our bounds. The philosophy is that if we have a martingale $M(t)$ and we want a bound, we need to control a modulus of continuity $\sup_{s\neq t, |s-t|<\delta} |M(t) - M(s)|$. Recall that if X is a centered Gaussian, $\mathbb{E}[e^{\lambda X}] = e^{(\lambda^2/2)\,\mathbb{E}[X^2]}$.

Proposition 1.1. *If we set $X_i = B(t_i^n) - B(s)$, then*

$$\mathbb{E}\left[\exp\left(\lambda \sum_{i=k}^{r-1} X_i B(t_i^n, t_{i+1}^n) - \frac{\lambda^2}{2} X_i^2 (t_{i+1}^n - t_i^n)\right)\right] = 1.$$

Proof.

$$\text{LHS} = \mathbb{E}\left[\exp\left(\lambda \sum_{i=k}^{r-2} X_i B(t_i^n, t_{i+1}^n) - \frac{\lambda^2}{2} X_i^2 (t_{i+1}^n - t_i)\right) e^{\lambda X_{r-1} B(t_{r-1}^n, t_r^n) - \frac{\lambda^2}{2} X_{r-1}^2 (t_r^n - r_{r-1}^n)}\right]$$

Condition on the past up to time t^n_{r-1}. The term on the right just becomes 1 because $B(t^n_{r-1}, t^n_r)$ is the only randomness.

$$= \mathbb{E}\left[\exp\left(\lambda \sum_{i=k}^{r-2} X_i B(t_i^n, t_{i+1}^n) - \frac{\lambda^2}{2} X_i^2 (t_{i+1}^n - t_i)\right)\right]$$

We can do the same thing, picking off one term at a time

$$= \cdots$$

$$= 1.$$

□

Remark 1.1. We may write this as

$$\mathbb{E}[e^{\lambda M_n - \frac{\lambda^2}{2} Z_n}] = 1,$$

where M_n is a martingale, and Z_n is the quadratic variation of M_n.

We wish to expand this expression in λ:

$$1 = \mathbb{E}\left[\sum_{m=0}^{\infty} K_m(M_n, Z_n) \frac{\lambda^m}{m!}\right].$$

From this we want to deduce that $K_0 = 1$ and $\mathbb{E}[K_m(M_n, Z_n)] = 0$ for all $m \geq 1$. This gives nice control on M_n in terms of its quadratic variation Z_n. Indeed, use the expansion:

$$e^{tx - \frac{t^2}{2}} = \sum_{m=0}^{\infty} (\mathrm{He})_m(x) \frac{t^m}{m!},$$

Where $(\mathrm{He})_m(x)$ is the m-th Hermite polynomial. Hermite polynomials satisfy the recursive identity $(\mathrm{He})_{m+1}(x) = x(\mathrm{He})_m(x) - m(\mathrm{He})_{m-1}(x)$. We also have $(\mathrm{He})_m(x) = 1$ and $(\mathrm{He})_1(x) = x$, so it is possible to show that $(\mathrm{He})_m(0) = 0$ if m is odd. We can also show that $(\mathrm{He})_m$ has even powers if m is even and odd powers if m is odd. Moreover, we have the expansion (setting $t = \lambda\sqrt{Z}$ and $x = \frac{M}{\sqrt{Z}}$)

$$e^{\lambda M - \frac{\lambda^2}{2} Z} = \sum_{m=0}^{\infty} K_m(M, Z) \frac{\lambda^m}{m!}, \qquad K_m(M, Z) = (\mathrm{He})_m \left(\frac{M}{\sqrt{Z}}\right) (\sqrt{Z})^m.$$

Observe that

$$K_{2m}(M, Z) = M^{2m} + c_1^m M^{2m-2} Z + \cdots + c_{m-1}^m M^2 Z^{m-1} + c_m^m Z^m.$$

From this an $\mathbb{E}[K_{2m}(M, Z)] = 0$, we learn that

$$\mathbb{E}[M^{2m}] \leq -\sum_{i=1}^{m} c_i^m \, \mathbb{E}[M^{2m-2i} Z^i].$$

Let's Schwarz this![1] Use the weighted Schwarz inequality, $ab \le \frac{(\varepsilon a)^p}{p} + \frac{(b/\varepsilon)^q}{q}$ to write

$$\begin{aligned}\mathbb{E}[M^{2m-2i}Z^i] &\le \frac{2m-2i}{2m}(\varepsilon M^{2m-2i})^{2m/(2m-2i)} + (Z^i/\varepsilon)^{m/i}\frac{i}{m} \\ &= \left(1-\frac{i}{m}\right)\varepsilon^{m/(m-i)}M^{2m} + \left(\frac{1}{\varepsilon}\right)^{m/i}\frac{i}{m}Z^m.\end{aligned}$$

From this, we deduce

$$\mathbb{E}[M^{2m}] \le c_m\,\mathbb{E}[Z^m].$$

In summary, if

$$M = M_n = \sum_{t_i^n \in [s,t]} (B_j(t_i^n) - B_j(s))B_k(t_i^n, t_{i+1}^n), \qquad B = (B_1, \dots, B_\ell),$$

then

$$\mathbb{E}[M_n] \le c_m\,\mathbb{E}[Z_n^m],$$

where

$$Z_n = \sum_{t_i^n} B_j(s, t_i^n)^2 (t_{i+1}^n - t_i^n).$$

Recall that if $\alpha \in (0, 1/2)$ and if

$$C(B) = \sup_{\substack{s \ne t \\ s,t \in [0,T]}} \frac{|B(s,t)|}{|t-s|^\alpha},$$

then $\mathbb{E}[C(B)^q] < \infty$ for every $q \ge 1$ (and in fact even $\mathbb{E}[e^{c_0 C(B)}] < \infty$). Then

$$\mathbb{E}[Z_n^m] \le \mathbb{E}[C(B)^m |t-s|^{2\alpha m + m}] \le c'_m |t-s|^{2\alpha m + m}.$$

As a result,

$$\left(\mathbb{E}[M_n^{2m}]\right)^{1/(4m)} \le c'_m c_m |t-s|^{(2\alpha+1)/4}.$$

In other words,

$$\|\sqrt{M_n}\|_{L^{4m}(\mathbb{P})} \le c|t-s|^{(2\alpha+1)/4},$$

and by Kolmogorov's theorem,

$$\mathbb{E}\left[\sup_{\substack{s \ne t \\ s,t \in [0,T]}} \frac{|\sqrt{M}_n(s,t)|}{|t-s|^\gamma}\right] < \infty,$$

provided that $\gamma \in (0, \frac{2\alpha+1}{4} - \frac{1}{4m})$. By choosing m large and α close to $1/2$, we can get any $\gamma \in (0, 1/2)$. Thus, we do have a rough path $(B, \mathbb{B})$ in $\mathscr{R}^\gamma$ with $\gamma \in (0, 1/2)$. Since $\widehat{\mathbb{B}}(s,t) = \mathbb{B}(s,t) - \frac{t-s}{2}I$, the same is true for $\widehat{\mathbb{B}}$.

[1] Maybe we shouldn't be using Schwarz as a verb, but this is how verbs are created.

1.2 Geometricity of the Stratonovich lift

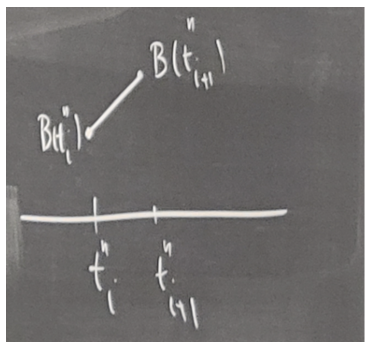
We now claim that $\widehat{\mathbb{B}}$ is geometric and that a smooth approximation of B would lead to the Stratonovich integration. Recall that we want to solve an equation like $\dot{y} = b(y) + \sigma(y)\dot{B}$; we have two candidates for the integrals in the corresponding integral equation, as well. If we replace B by a smooth approximation $B_\varepsilon \xrightarrow{\varepsilon \to 0} B$, then we can solve the equation $\dot{y}_\varepsilon = b_\varepsilon(y) + \sigma_\varepsilon(y)\dot{B}_\varepsilon$ classically. Then $\lim_{\varepsilon \to 0} y_\varepsilon = y$, so

$$\dot{y} = b(y) + \sigma(y)\frac{d}{dt}\widehat{B}.$$

Thus, it will be the Stratonovich integral, not the Itô integral. Note that the regularization should be independent of the path.

To explain this, let us observe that if B is a Brownian motion and $B^{(n)}$ is the linear interpolation

$$B^{(n)}(t) = \sum_{i=0}^{\infty} \mathbb{1}_{[t_i^n, t_{i+1}^n]}(t) \cdot \left[\frac{t - t_i^n}{t_{i+1}^n - t_i^n} B(t_{i+1}^n) + \frac{t_{i+1}^n - t}{t_{i+1}^n - t_i^n} B(t_i^n)\right],$$



then

$$\begin{aligned}
\int_s^t B^{(n)}(\theta) \otimes dB^{(n)}(\theta) &= \sum_{t_i^n \in [s,t]} \int_{t_i^n}^{t_{i+1}^n} B^{(n)}(\theta) \otimes dB^{(n)}(\theta) \\
&= \sum_{t_i^n \in [s,t]} (t_{i+1}^n - t_i^n) \frac{B(t_i^n) + B(t_{i+1}^n)}{2} \otimes \frac{B(t_i^n, t_{i+1}^n)}{t_{i+1}^n - t_i^n} \\
&= \text{Stratonovich approximation.}
\end{aligned}$$

So for $\alpha \in (0, 1/2)$,

$$\left(B^{(n)}, \int B^n \otimes dB^{(n)}\right) \xrightarrow{\mathscr{R}^\alpha} (B, \widehat{\mathbb{B}})$$

because we already know the L^2-convergence, and we have established a uniform bound on $\mathscr{R}^\alpha$ of the approximation. Hence, we have convergence in $\mathscr{R}^\beta$ for $\beta < \alpha$.

Remark 1.2. We can have the following probabilistic interpretation for our approximation that offers another proof of the L^2-convergence. Namely, if $\mathcal{F}_n$ is the σ-algebra generated by $(B(t_i^n) : i = 0, 1, 2, \ldots)$, then $B^{(n)} = \mathbb{E}[B \mid \mathcal{F}_n]$. Then $B^{(n)} \to B$ follows from the celebrated Doob's martingale convergence theorem.